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# Semi-Lipschitz Functions and Best Approximation in Quasi-Metric Spaces[1]

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We show that the set of semi-Lipschitz functions, defined on a quasi-metric space $(X, d)$, that vanish at a fixed point $x_0 \in X$ can be endowed with the structure of a quasi-normed semilinear space. This provides an appropriate setting in which to characterize both the points of best approximation and the semi-Chebyshev subsets of quasi-metric spaces. We also show that this space is bicomplete. © 2000 Academic Press

*Key Words:* semi-Lipschitz function; best approximation; quasi-norm; quasi-metric; bicomplete.

## 1. INTRODUCTION AND BASIC RESULTS

Motivated, in part, for their applications to computer science (see for instance [16, 27, 30]), the theories of completeness, (pre)compactness, and extension of quasi-uniform and quasi-metric spaces have received a certain attention in the recent years (see, among other contributions, [1, 2, 4, 14, 24, 31, 32]). These advances have also permitted the development of

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generalizations, to the "nonsymmetric case," of classical mathematical theories: hyperspaces [15, 22, 33], function spaces [13, 20, 21, 23, 26], approximation and fixed point theory [6, 7, 25], linear lattices [3], etc.

This paper is a contribution to the study of semi-Lipschitz functions and best approximation from a nonsymmetric point of view. In searching for the category of domains of computation, quasi-metric spaces, along with certain closely related categories, are worthy of consideration. Usually, a metric gives us no means of expressing the ordering of information; while, if all we have is the ordering, only qualitative distinctions can be expressed. A way out of this dilemma is to move to quasi-metric (or quasi-uniform) spaces. In addition, computer science provides an abundance of examples which can be of help in obtaining the theory of limits and completeness for quasi-metric and quasi-uniform spaces (see, for example, [28, 29]). Given a quasi-metric space we introduce and study a type of generalized Lipschitz functions. In particular, we show that the family of such functions admits a structure of bicomplete quasi-normed space. In the final part of the paper we show how this structure provides an appropriate setting to characterize both the points of best approximation and the semi-Chebyshev subsets of a quasi-metric space. In this way our results generalize the metric theory of Lipschitz functions [8] and best approximation [17–19].

In this paper, *a quasi-metric* on a set $X$ will be a function $d: X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$,

(ii) $d(x, y) \leqslant d(x, z) + d(z, y)$.

If $d$ can take the value $\infty$ then it is called *an extended quasi-metric* on $X$. Each (extended) quasi-metric $d$ induces another (extended) quasi-metric $d^{-1}$ (defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$) called the conjugate of $d$. Therefore the function $d^s$ defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a(n) (extended) metric on $X$.

By (*extended*) *quasi-metric space* we mean a pair $(X, d)$ where $X$ is a set and $d$ is a(n) (extended) quasi-metric on $X$.

Each extended quasi-metric $d$ on $X$ induces a topology $T(d)$ on $X$ which has as a base the family of balls $\{B_d(x, r): x \in X, r > 0\}$ where $B_d(x, r) = \{y \in X : d(x, y) < r\}$. A topological space $(X, T)$ is called *quasi-metrizable* if there is a quasi-metric $d$ on $X$ such that $T(d) = T$. In this case we say that $d$ is compatible with $T$. We remark that the topology $T(d)$ is $T_0$. Moreover, if condition (i) above is replaced by the condition $d(x, y) = 0 \Leftrightarrow x = y$, then $T(d)$ is a $T_1$ topology. In this case the pair $(X, d)$ is said to be *a*(*n*) (*extended*) $T_1$ *quasi-metric space*. See [5] for more information about quasi-metric spaces. A(n) (extended) quasi-metric $d$ is said to be bicomplete if $d^s$ is a complete (extended) metric.

A *semilinear space* (or *semivector space*) on $\mathbf{R}^+$ (the set of all positive real numbers) is a triple $(X, +, \cdot)$ such that $(X, +)$ is an Abelian semigroup with neutral element $0 \in X$ and $\cdot$ is a function from $\mathbf{R}^+ \times X$ into $X$ which satisfies for all $x, y \in X$ and $a, b \in \mathbf{R}^+$: (i) $a \cdot (b \cdot x) = (ab) \cdot x$, (ii) $(a+b) \cdot x = (a \cdot x) + (b \cdot x)$, (iii) $a \cdot (x+y) = (a \cdot x) + (a \cdot y)$, and (iv) $1 \cdot x = x$. (See [11] for related structures). As usual, whenever an element $x \in X$ admits an inverse it is unique and is denoted $-x$.

According to [3] *a quasi-norm* on a *semilinear space* $(X, +, \cdot)$ on $\mathbf{R}^+$ is a function $\|.\|: X \to [0, \infty)$ such that for all $x, y \in X$ and $a \in \mathbf{R}^+$: (i) $x = 0 \Leftrightarrow -x \in X$ and $\|x\| = \|-x\| = 0$, (ii) $\|a \cdot x\| = a\,\|x\|$, and (iii) $\|x+y\| \leqslant \|x\| + \|y\|$. The pair $(X, \|.\|)$ is then called *a quasi-normed semilinear space*.

Let $(X, d)$ be a quasi-metric space. A function $f: X \to \mathbf{R}$ is said to be *semi-Lipschitz* if there exists $k \geqslant 0$ such that $f(x) - f(y) \leqslant k d(x, y)$ for all $x, y \in X$. Clearly, every semi-Lipschitz function is lower semicontinuous.

A real-valued function $f$ defined on a quasi-metric space $(X, d)$ is said to be $\leqslant_d$*-increasing* if $f(x) \leqslant f(y)$ whenever $d(x, y) = 0$. Note that every semi-Lipschitz function on $(X, d)$ is $\leqslant_d$-increasing and every real-valued function defined on a $T_1$ quasi-metric space $(X, d)$ is $\leqslant_d$-increasing.

Now let $(X, d)$ be a quasi-metric space and fix $x_0 \in X$. Put

$$\mathcal{SL}_0(d) =$$

$$\left\{ f: X \to \mathbf{R} \mid f \text{ is } \leqslant_d\text{-increasing}, \sup_{d(x, y) \neq 0} \frac{(f(x) - f(y)) \vee 0}{d(x, y)} < \infty, f(x_0) = 0 \right\}.$$

It is straightforward to see that $\mathcal{SL}_0(d)$ is exactly the set of all semi-Lipschitz functions on $(X, d)$ that vanish at $x_0$.

If for all $f, g \in \mathcal{SL}_0(d)$ and $a \in \mathbf{R}^+$ we define $f+g$ and $a \cdot f$ in the usual way, then it is routine to show that $(\mathcal{SL}_0(d), +, \cdot)$ is a semilinear space on $\mathbf{R}^+$.

Furthermore, the function $\|.\|_d$ defined on $\mathcal{SL}_0(d)$ by

$$\|f\|_d = \sup_{d(x, y) \neq 0} \frac{(f(x) - f(y)) \vee 0}{d(x, y)}$$

is clearly a quasi-norm on $\mathcal{SL}_0(d)$, so that $(\mathcal{SL}_0(d), \|.\|_d)$ is a quasi-normed semilinear space.

In addition, the function $\rho_d$ defined on $\mathcal{SL}_0(d) \times \mathcal{SL}_0(d)$ by

$$\rho_d(f, g) = \sup_{d(x, y) \neq 0} \frac{((f-g)(x) - (f-g)(y)) \vee 0}{d(x, y)}$$

is an extended quasi-metric on $\mathcal{SL}_0(d)$. Notice that $\rho_d(f, g)$ agrees with $\|f-g\|_d$ whenever $f-g \in \mathcal{SL}_0(d)$.

The following example shows that the Abelian semigroup $(\mathcal{SL}_0(d), +)$, which has a neutral element, is not in general a group. Moreover $\rho_d$ is not necessarily a quasi-metric.

EXAMPLE 1.1. Let $d$ be the $T_1$ quasi-metric defined on $\mathbf{R}$ by $d(x, y) = x - y$ if $x \geqslant y$ and $d(x, y) = 1$ otherwise. Then $T(d)$ is the Sorgenfrey line. Let $x_0 = 0$. If we denote the identity function on $\mathbf{R}$ by $id$, then $id \in \mathcal{SL}_0(d)$ because $\sup_{x \neq y} \frac{(x-y) \vee 0}{d(x, y)} = 1$. However, $-id \notin \mathcal{SL}_0(d)$ because $\sup_{x \neq y} \frac{(y-x) \vee 0}{d(x, y)} = \infty$. So $\rho_d(\mathbf{0}, id) = \infty$, where $\mathbf{0}$ denotes the function on $\mathbf{R}$ that vanishes at every $x \in \mathbf{R}$.

Several properties of the quasi-normed spaces of semi-Lipschitz functions (completeness and compactness, characterization for it to be a Banach space, etc.) will be discussed elsewhere. However, in order to point out the analogy between the symmetric case and the nonsymmetric one, we include here a basic property on completeness of $\rho_d$ which may be considered as an analogous result to the corresponding theorem for the normed linear space of Lipschitz functions on a metric space, obtained by Johnson in [8].

THEOREM 1.2. *$\rho_d$ is a bicomplete extended quasi-metric on $\mathcal{SL}_0(d)$.*

*Proof.* Let $\{f_n\}_{n<\omega}$ be a Cauchy sequence in $(\mathcal{SL}_0(d), \rho_d^s)$. Then, given $\varepsilon > 0$ there is $n_0 < \omega$ such that $\rho_d(f_n, f_m) < \varepsilon$ and $\rho_d(f_m, f_n) < \varepsilon$ for $m, n \geqslant n_0$. Therefore

$$\sup_{d(x, y) \neq 0} \frac{|(f_n - f_m)(x) - (f_n - f_m)(y)|}{d(x, y)} < \varepsilon, \qquad \text{for all} \quad n, m \geqslant n_0. \qquad (*)$$

Then $|f_n(x) - f_m(x)| < \varepsilon d^s(x, x_0)$ for all $x \in X$ and for all $n, m \geqslant n_0$.

It follows that there is a function $f\colon X \to \mathbf{R}$ such that the sequence $\{f_n\}_{n<\omega}$ is pointwise convergent to $f$ with respect to the usual metric on $\mathbf{R}$. We shall show that $f$ is in $\mathcal{SL}_0(d)$ and that $\{f_n\}_{n<\omega}$ converges to $f$ with respect to the topology $T(\rho_d^s)$. For this in turn, first notice that, $f$ being the pointwise limit of a sequence of $\leqslant_d$-increasing functions, it is also $\leqslant_d$-increasing. On the other hand, since the sequence $\{(f_n(x) - f_n(y))/d(x, y)\}_{n<\omega}$ converges to $(f(x) - f(y))/d(x, y)$ with respect to the usual metric on $\mathbf{R}$ whenever $d(x, y) \neq 0$, for the given $\varepsilon > 0$, for $n \geqslant n_0$ and for $x, y \in X$ with $d(x, y) \neq 0$, there exists $m \geqslant n$ such that

$$\frac{|(f - f_m)(x) - (f - f_m)(y)|}{d(x, y)} < \varepsilon.$$

So, by the triangle inequality and condition $(*)$

$$\frac{|(f-f_n)(x)-(f-f_n)(y)|}{d(x, y)} < 2\varepsilon \qquad (**)$$

whenever $n \geqslant n_0$ and $d(x, y) \neq 0$. It follows from $(**)$ that

$$\sup_{d(x, y) \neq 0} \frac{(f(x)-f(y)) \vee 0}{d(x, y)} \leqslant 2\varepsilon + \|f_{n_0}\|_d.$$

So $f \in \mathcal{SL}_0(d)$. Finally, it also follows from $(**)$ that the sequence $\{\rho_d^s(f, f_n)\}_{n<\omega}$ converges to zero and the proof is complete. ∎

## 2. ON BEST APPROXIMATION IN QUASI-METRIC SPACES

Let $(X, d)$ be a quasi-metric space and let $y \in Y \subset X$. We shall denote by $cl_X\{y\}$ the closure $\{x\colon d(x, y)=0\}$ of the subset $\{y\}$ in the topology $T(d)$. As usual, $d(p, Y)$ shall denote $\inf\{d(p, y) \mid y \in Y\}$ for each $p \in X$.

Definition 2.1. Let $(X, d)$ be a quasi-metric space. Let $Y \subset X$ and $p \in X$. An element $y_0 \in Y$ such that $d(p, Y)=d(p, y_0)$ is said to be an element of best approximation to $p$ by elements of $Y$.

Note that if $d(p, y_0)=0$ for some $y_0 \in Y$ then $y_0$ is obviously a *trivial* element of best approximation to $p$ by elements of $Y$. Therefore we focus our interest on those points $p \notin \bigcup\{cl_X\{y\} \mid y \in Y\}$.

Proposition 2.2. *Let $(X, d)$ be a quasi-metric space. Let $Y \subset X$, $x_0 \in Y$ and $p \notin \bigcup\{cl_X\{y\} \mid y \in Y\}$. Then $y_0 \in Y$ is an element of best approximation to $p$ by elements of $Y$ if and only if there is an $f \in \mathcal{SL}_0(d)$ such that*

(1) $\|f\|_d = 1$,

(2) $f_{|Y} = 0$,

(3) $d(p, y_0) = f(p) - f(y_0)$.

*Proof.* If $p \notin \bigcup\{cl_X\{y\} \mid y \in Y\}$ and $y_0 \in Y$ is an element of best approximation to $p$ by elements of $Y$ define $f\colon X \to \mathbf{R}$ by $f(x)=d(x, Y)$. Then $f_{|Y}=0$. On the other hand, given two points $x, z \in X$ with $d(x, z)=0$, the triangle inequality say us that $d(x, y) \leqslant d(z, y)$ for each $y \in Y$, that is, $d(x, Y) \leqslant d(z, Y)$. Thus, $f$ is a $\leqslant_d$-increasing function. In addition,

$$\sup_{d(x, y) \neq 0} \frac{(f(x) - f(y)) \vee 0}{d(x, y)} = \sup_{d(x, y) \neq 0} \frac{(d(x, Y) - d(y, Y)) \vee 0}{d(x, y)} \leqslant \sup_{d(x, y) \neq 0} \frac{d(x, y)}{d(x, y)},$$

so that $f \in \mathcal{SL}_0(d)$ and $\|f\|_d \leqslant 1$. We shall show that actually $\|f\|_d = 1$. In fact, since $f(p) - f(y_0) = d(p, Y) = d(p, y_0)$ and $d(p, y_0) > 0$,

$$\frac{(f(p) - f(y_0)) \vee 0}{d(p, y_0)} = \frac{d(p, Y)}{d(p, y_0)} = 1,$$

so that $\|f\|_d \geqslant 1$.

Conversely, for each $y \in Y$,

$$d(p, y) = \|f\|_d \, d(p, y) \geqslant \frac{(f(p) - f(y)) \vee 0}{d(p, y)} \, d(p, y).$$

Therefore, for each $y \in Y$,

$$d(p, y) \geqslant f(p) - f(y) = f(p) - f(y_0) = d(p, y_0)$$

by (3), which proves that $y_0$ is an element of best approximation to $p$ by elements of $Y$. ∎

Let $(X, d)$ be a quasi-metric space, $Y \subset X$ and $x_0 \in Y$. Let

$$Y_0 = \{f : X \to \mathbf{R} \mid f \in \mathcal{SL}_0(d) \text{ and } f_{|Y} = 0\},$$

and let us define for each $x, y \in X$ such that $d(x, y) \neq 0$,

$$d_{Y_0}(x, y) = \sup \left\{ \frac{(f(x) - f(y)) \vee 0}{\|f\|_d} \,\middle|\, f \in Y_0 \text{ and } \|f\|_d \neq 0 \right\}.$$

Then $d_{Y_0}(x, y) \leqslant d(x, y)$. In fact, for all $f \in \mathcal{SL}_0(d)$, $(f(x) - f(y)) \vee 0 \leqslant \|f\|_d \, d(x, y)$ since $d(x, y) \neq 0$. Thus, $((f(x) - f(y)) \vee 0)/\|f\|_d \leqslant d(x, y)$ for $f \in \mathcal{SL}_0(d)$, $\|f\|_d \neq 0$. Hence,

$$d_{Y_0}(x, y) \leqslant \sup \left\{ \frac{(f(x) - f(y)) \vee 0}{\|f\|_d} \,\middle|\, f \in \mathcal{SL}_0(d) \text{ and } \|f\|_d \neq 0 \right\} \leqslant d(x, y).$$

We now have the following result.

PROPOSITION 2.3. *Let $(X, d)$ be a quasi-metric space. Let $Y \subset X$, $x_0 \in Y$ and $p \notin \bigcup\{cl_X\{y\} \mid y \in Y\}$. Then $y_0 \in Y$ is an element of best approximation to $p$ by elements of $Y$ if and only if $d_{Y_0}(p, y_0) = d(p, y_0)$.*

*Proof.* Let $y_0 \in Y$ be an element of best approximation to $p$ by elements of $Y$. By Proposition 2.1 there is $f \in Y_0$ such that $\|f\|_d = 1$ and $d(p, y_0) = f(p) - f(y_0)$. Therefore

$$d_{Y_0}(p, y_0) = \sup\left\{\frac{(g(p) - g(y_0)) \vee 0}{\|g\|_d} \;\middle|\; g \in Y_0,\ \|g\|_d \neq 0\right\}$$
$$\geqslant \frac{(f(p) - f(y_0)) \vee 0}{\|f\|_d} = d(p, y_0).$$

Since $d_{Y_0}(p, y_0) \leqslant d(p, y_0)$, we conclude that $d_{Y_0}(p, y_0) = d(p, y_0)$.

Conversely, for all $y \in Y$,

$$d(p, y_0) = d_{Y_0}(p, y_0) = \sup\left\{\frac{(f(p) - f(y_0)) \vee 0}{\|f\|_d} \;\middle|\; f \in Y_0,\ \|f\|_d \neq 0\right\}$$
$$= \sup\left\{\frac{(f(p) - f(y)) \vee 0}{\|f\|_d} \;\middle|\; f \in Y_0,\ \|f\|_d \neq 0\right\} = d_{Y_0}(p, y) \leqslant d(p, y),$$

so that $y_0 \in Y$ is an element of best approximation to $p$ by elements of $Y$. ∎

Let $Y$ be a (nonempty) subset of a quasi-metric space $(X, d)$. For each $p \notin Y$ we shall denote by $P_Y(p)$ the set of all best approximation to $p$ by elements of $Y$. A (nonempty) set $Y \subset X$ such that $X \setminus \bigcup\{cl_X\{y\} \mid y \in Y\} \neq \emptyset$ is said to be *semi-Chebyshev* if card $P_Y(p) \leqslant 1$ for each $p \notin \bigcup\{cl_X\{y\} \mid y \in Y\}$.

PROPOSITION 2.4. *Let $Y$ be a (nonempty) subset of a quasi-metric space $(X, d)$. Let $M \subset Y$, $x_0 \in Y$ and $p \notin \bigcup\{cl_X\{y\} \mid y \in Y\}$. Then $M \subset P_Y(p)$ if and only if there is an $f \in \mathcal{SL}_0(d)$ such that*

(1) $\|f\|_d = 1$,

(2) $f_{|Y} = 0$,

(3) $d(p, y) = f(p) - f(y)$ *for all* $y \in M$.

*Proof.* Suppose $M \subset P_Y(p)$. Fix $y_0 \in M$. By Proposition 2.2 there exists $f \in \mathcal{SL}_0(d)$ satisfying (1), (2) and $d(p, y_0) = f(p) - f(y_0)$. Let $y \in M$. Then $d(p, y) = d(p, Y) = d(p, y_0)$, so $d(p, y) = f(p) - f(y_0)$. Since $f_{|Y} = 0$ we obtain that $d(p, y) = f(p) = f(p) - f(y)$.

Conversely, suppose that there exists $f \in \mathcal{SL}_0(d)$ satisfying (1), (2) and (3) and let $y_0 \in M$. By Proposition 2.2, $y_0 \in P_Y(p)$. Hence $M \subset P_Y(p)$. ∎

The next proposition follows easily from Proposition 2.3.

PROPOSITION 2.5. *Let $Y$ be a (nonempty) subset of a quasi-metric space $(X, d)$. Let $M \subset Y$, $x_0 \in Y$ and $p \notin Y$. Then $M \subset P_Y(p)$ if and only if $d_{Y_0}(p, y) = d(p, y)$ for all $y \in M$.*

As an immediate consequence of Proposition 2.4 we obtain the following characterization of semi-Chebyshev sets in a quasi-metric space (compare with [18, 19]).

PROPOSITION 2.6. *Let $(X, d)$ be a quasi-metric space. Let $Y \subset X$ and $x_0 \in Y$. Then $Y$ is semi-Chebyshev if and only if there does not exist $f \in \mathcal{SL}_0(d)$, $x_1 \in X$ and $y_1, y_2 \in Y$, $y_1 \neq y_2$, such that*

(1) $\|f\|_d = 1$,

(2) $f_{|Y} = 0$,

(3) $f(x_1) = d(x_1, y_1) = d(x_1, y_2)$.

In the following example we shall apply the above results in order to characterize the sets $P_Y(p)$ in the Khalimsky line. The *Khalimsky line* is an interesting quasi-metric space in digital topology (see [9, 10, 12]). The Khalimsky line is the set $\mathbf{Z}$ of integers endowed with the topology induced by the quasi-metric $d$ defined as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } \ x = y \\ 0 & \text{if } x = 2n \text{ and } y = 2n+1 \text{ or } y = 2n-1 \text{ where } n \in \mathbf{Z}, \\ 1 & \text{otherwise.} \end{cases}$$

It is an easy matter to see that for each $x_0 \in \mathbf{Z}$, $\mathcal{SL}_0(d)$ is exactly the set of all bounded $\leqslant_d$-increasing functions on $\mathbf{Z}$.

EXAMPLE 2.2. Let $(X, d)$ be the Khalimsky line. Let $Y \subset X$ and let $p \notin \bigcup\{cl_X\{y\} \mid y \in Y\}$. Fix $x_0 \in Y$ and define the function $f$ from $X$ into $\mathbf{R}$ by $f(x) = d(x, Y)$ for every $x \in X$. As in the proof of Proposition 2.2, $f \in \mathcal{SL}_0(d)$ and $\|f\|_d = 1$. Since $f_{|Y} \equiv 0$ and $d(p, y) = f(p) - f(y)$ whenever $y \in Y$, we conclude, by Proposition 2.4, that $Y = P_Y(p)$.

Next, if $p \notin Y$ but $p \in \bigcup\{cl_X\{y\} \mid y \in Y\}$, then $p$ is an even integer and $p+1$ or $p-1$ are in $Y$. Thus, $P_Y(p)$ is not empty and contains at most two points.

Notice that, as $Y = P_Y(p)$ whenever $p \notin \bigcup\{cl_X\{y\} \mid y \in Y\}$, the semi-Chebyshev subsets in the Khalimsky line are the singletons.

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